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2006 J. Phys. A: Math. Gen. 39 7789

(http://iopscience.iop.org/0305-4470/39/25/S01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7789–7799 doi:10.1088/0305-4470/39/25/S01

Some specific features of the ε expansion in the theory of turbulence and the possibility of its improvement

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Received 28 November 2005
Published 7 June 2006
Online at stacks.iop.org/JPhysA/39/7789

Abstract
Specific problems arising in the use of the method of renormalization group and ε expansion in the theory of developed turbulence are discussed: the necessity to take into account the dependence of the model parameters on the expansion parameter ε and the large magnitude of the physical value of the expansion parameter. It is shown that quantities independent of the amplitude of the correlation function of the random force possess a uniquely defined ε expansion. On the example of the two-loop calculation of the Kolmogorov constant and the turbulent Prandtl number, effectiveness of the improved expansion is demonstrated, which uses an approximate summation of the high-order terms of the ε expansion.

PACS numbers: 47.27.Ef, 47.27.Gs, 47.27.Te

1. Introduction

In the stochastic model of turbulence the eddy velocity field $\varphi_i(\mathbf{x}, t)$ of an incompressible fluid conforms to the Navier–Stokes equation with a random force

$$\partial_t\varphi_i + (\varphi_j\partial_j)\varphi_i = \nu_0\Delta\varphi_i - \partial_i P + F_i, \tag{1}$$

where $P(t, \mathbf{x})$ is the pressure and $F_i(t, \mathbf{x})$ is a transverse external random force per unit mass compensating for the losses of the eddy energy due to viscous dissipation and ensuring the existence of the turbulent stationary regime. For F a Gaussian distribution is assumed with zero mean and the correlation function

$$\langle F_i(x)F_j(x')\rangle = \delta(t-t')(2\pi)^{-d}\int \mathrm{d}\mathbf{k}\, P_{ij}(\mathbf{k})d_F(k)\exp[\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')], \tag{2}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projection operator, $d_F(k)$ is a function of $k \equiv |\mathbf{k}|$ and model parameters and d is the dimension of the $\mathbf{x}$ space. In the steady state, the integral of the 'pumping function' $d_F(k)$ coincides with the energy dissipation rate per unit mass $\bar{\mathcal{E}}$

$$\bar{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int \mathrm{d}\mathbf{k}\, d_F(k). \tag{3}$$

Objects to be investigated in the model (1), (2) are the correlation functions $\langle \varphi(\mathbf{x}_1, t_1), \varphi(\mathbf{x}_2, t_2) \cdots \varphi(\mathbf{x}_n, t_n) \rangle$, where the brackets $\langle \cdots \rangle$ denote averaging over the random force F. There is extensive experimental information about the structure functions $S_n(r)$—single-time correlation functions of the form

$$S_n(r) \equiv \langle [\varphi_r(t, \mathbf{x}+\mathbf{r}) - \varphi_r(t, \mathbf{x})]^n \rangle, \qquad \varphi_r \equiv \frac{(\varphi_i \cdot r_i)}{|\mathbf{r}|}. \tag{4}$$

According to the experimental data the functions $S_n(r)$ assume the simplest form in the inertial range $r_{\mathrm{diss}} \ll r \ll L$

$$S_n(r) \simeq C_n (\bar{\mathcal{E}} r)^{n/3} (L/r)^{\xi_n}, \qquad r_{\mathrm{diss}} \ll r \ll L, \tag{5}$$

where $r_{\mathrm{diss}} \equiv (\nu^3/\bar{\mathcal{E}})^{1/4}$ is the characteristic size of the dissipating eddies. The existence of the inertial range in systems with developed turbulence (large Reynolds numbers $Re \gg 1$) is ensured by the inequality $L/r_{\mathrm{diss}} \sim Re^{3/4} \gg 1$. The task of the theory is to substantiate the powerlike asymptotics (5) and calculate the exponents ξ_n and amplitudes C_n.

At present there is a rigorous result only for the function $S_3(r)$:

$$S_3(r) = -\frac{12}{d(d+2)} \bar{\mathcal{E}} r, \tag{6}$$

which corresponds to values $C_3 = -12/d(d+2)$ and $\xi_3 = 0$ in (5), obtained by Kolmogorov on the basis of the analysis of the equation of the spectral balance of energy [1]. In accordance with the theory of Kolmogorov the function $S_3(r)$ in the inertial range does not depend on either the viscosity (or on r_{diss}) or on the external scale L. According to (5) independence of the viscosity holds for all functions $S_n(r)$. However, as the experiment (available to $n = 18$) shows, the second assumption of Kolmogorov is not fulfilled in the general case: the exponents ξ_n are nonzero and monotonically grow with the growth of n, i.e. the functions $S_n(r)$ in the inertial range keep a 'memory' of the external scale of turbulence ('anomalous scaling'). The problem of substantiation of the anomalous scaling and calculation of the anomalous exponents ξ_n has not been solved up to date; significant progress in this respect, however, has been made in the related problem of turbulent mixing of a passive admixture (see, e.g., [2]).

Among the structure functions $S_n(r)$ the functions $S_2(r)$ and $S_3(r)$ connected with the spectral density of the pulsation energy ($S_2(r)$) and its transfer velocity ($S_3(r)$) have a special place. As noted already, for $S_3(r)$ hypotheses of the theory of Kolmogorov are completely fulfilled. As to the anomalous exponent ξ_2, there is no unambiguous experimental proof of its deviation from zero, it may only be asserted that ξ_2 is small. Thus, the experimental data are fairly accurately described by the known 'two thirds' law of Kolmogorov

$$S_2(r) \simeq C_{\mathrm{K}} (\bar{\mathcal{E}} r)^{2/3}, \qquad r_{\mathrm{diss}} \ll r \ll L, \tag{7}$$

where C_{K} is the Kolmogorov constant (see the discussion in [3]). We also note that in the model of turbulent transport of the passive admixture it has been proved that the anomalous scaling is absent in the second-order structure function ($\xi_2 = 0$) [2].

The stochastic problem (1)–(2) is equivalent to the quantum-field model with the number of fields doubled $\phi \equiv \{\varphi, \varphi', \}$ and with the action

$$S(\Phi) = \varphi' D_F \varphi'/2 + \varphi'[-\partial_t \varphi + \nu_0 \Delta\varphi - (\varphi\partial)\varphi], \tag{8}$$

where D_F is the correlation function of the random force (2), and necessary integration over $\{t, \mathbf{x}\}$ and summation over vector indices are implied. The equivalence to the stochastic problem (1)–(2) means that the averages $\langle\cdots\rangle$ over the distribution of the random force F coincide with the corresponding functional averages over the pair of fields $\phi \equiv \{\varphi, \varphi'\}$ with the weight $\sim \exp S(\phi)$.

To apply the renormalization-group (RG) method, it is necessary to use in (2) a pumping function $d_F(k)$ of a special form [4, 5]

$$d_F(k) = D_0 k^{4-d-2\varepsilon}. \tag{9}$$

In the infrared region the power function (9) is assumed to be cut off at $k \leqslant m \equiv L^{-1}$. The quantity $\varepsilon > 0$ in (9) in the renormalization-group approach plays the role of the formal small expansion parameter, whose physical value is considered to be $\varepsilon = 2$ (see below for details).

The model (2), (8) gives rise to the standard perturbation theory and Feynman's diagrammatic technique with the charge $g_0 \equiv D_0 \big/ \nu_0^3$ as the expansion parameter. On dimensional grounds $g_0 \sim \mu^{2\varepsilon}$ (μ is the renormalization mass), wherefrom it is seen that—contrary to the theory of critical phenomena—in the theory of turbulence it is impossible to achieve the logarithmic model (dimensionless g_0) with the use of the space dimension d, but only putting $\varepsilon = 0$. Another important difference between these theories is that the amplitude D_0 in (2) depends on ε in an essential way. Let us explain this in more detail.

The pumping function $d_F(k)$ is not universal. On physical grounds it may only be asserted that it is localized on scales $k \sim m \equiv L^{-1}$, where L is the external scale of turbulence (the size of eddies pumped in the system). If we are interested in eddies of significantly lesser size (inertial and dissipation ranges), then the pumping function $d_F(k)$ may be considered proportional to the δ function $\delta(\mathbf{k})$. We obtain a powerlike model of such a δ function from (9), if we suppose that the limit $\varepsilon \to 2 - 0$ with the amplitude $D_0 \sim (2 - \varepsilon)$ corresponds to the physical case. This is corroborated by the explicit form of the structure function $S_3(r)$ in the inertial range, for which in the model (2), (8), (9) the generalization of the Kolmogorov relation (6) holds [6]

$$S_3(r) = -\frac{3(d-1)\Gamma(2-\varepsilon)(r/2)^{2\varepsilon-3} D_0}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon)}. \tag{10}$$

For the relation (10) to yield (6) in the limit $\varepsilon \to 0$ it is necessary that

$$D_0 \simeq \frac{4(2-\varepsilon)\Lambda^{2\varepsilon-4}}{\overline{S}_d(d-1)}\bar{\mathcal{E}}, \qquad \varepsilon \to 2 - 0, \tag{11}$$

which just corresponds to the powerlike model of the δ function described above (here, Λ is an arbitrary parameter of the dimension of inverse length and $\overline{S}_d = S_d/(2\pi)^d$, where S_d is the area of the d-dimensional unit sphere).

The diagrams of the perturbation theory diverge at $\varepsilon = 0$. In the limit $\varepsilon \to +0$ these divergences appearing in the form of poles in ε are removed by the multiplicative renormalization of the viscosity ν_0 and the charge $g_0 = D_0 \big/ \nu_0^3$

$$\begin{aligned} \nu_0 &= \nu Z_\nu, & D_0 &= g_0 \nu_0^3 = g\mu^{2\varepsilon}\nu^3, \\ g_0 &= g\mu^{2\varepsilon} Z_g, & Z_g &= Z_\nu^{-3} \end{aligned} \tag{12}$$

with the only independent renormalization constant Z_ν—a consequence of the Galilei invariance of the theory and absence of divergence at $d > 2$ of the non-local term $\sim \varphi'\varphi'$ in the action (8).

On the basis of the renormalization constants, the β functions are found in the standard fashion and the position of the fixed point determined $\beta(g_*) = 0$. One-loop calculation shows that such a fixed point $g_* \sim \varepsilon$ does exist and that it is IR stable: $\beta'(g_*) > 0$. The connection

(12) between the renormalization constants has the consequence that at this fixed point the critical dimension Δ_φ of the field φ is determined exactly: $\Delta_\varphi = 1 - 2\varepsilon/3$. From this point of view, calculation of the diagrams is required only for a confirmation of the stability of the fixed point. For the structure functions $S_n(r)$ it gives rise to the following IR asymptotics:

$$S_n(r) \simeq r^{-n\Delta_\varphi} R_n(r/L) = r^{(2\varepsilon/3-1)n} R_n(r/L), \qquad r \gg r_{\mathrm{diss}}. \tag{13}$$

The renormalization-group representation (13) corroborates the hypothesis of Kolmogorov of independence of the structure functions of the viscosity in the IR region. The second hypothesis of independence of $S_n(r)$ of the external scale L in the inertial range is tantamount to assumption that the limiting values $R_n(r/L \to 0)$ are finite, whereas the anomalous scaling (5) is obtained, if $R_n(r/L) \sim (r/L)^{-\xi_n}$, when $r/L \to 0$.

The RG approach allows us to calculate the functions $R_n(r/L)$ as an expansion in the formal small parameter ε. The perturbation theory in such a form cannot yield relations of the anomalous scaling. Indeed, let us assume that some exponent ξ of the anomalous scaling possesses an ε expansion $\xi = \xi^{(0)} + \varepsilon\xi^{(1)} + \cdots$. Then in the renormalization-group calculation of the function $R_n(r/L)$ the factor $(r/L)^{-\xi}$ turns out to have a representation as the expansion $(r/L)^{-\xi} = (r/L)^{-\xi^{(0)}}[1 - \xi^{(1)}\varepsilon \ln(r/L) + \cdots]$ and for the correct investigation of the asymptotics $r/L \to 0$ it is necessary to sum the 'leading logarithms' $\varepsilon^n \ln^n(r/L)$ and to prove that they actually are exponentiated into corresponding powers. Such a summation is not carried out directly by the renormalization group (it sums 'ultraviolet leading logarithms' of the form $\varepsilon^n \ln^n(r/r_{\mathrm{diss}})$). In the theory of critical phenomena, the analogous problem is solved with the aid of operator expansion, which provides the proof of the powerlike character of the asymptotics discussed, with the interpretation of the exponents $-\xi_n$ as dimensions of some composite operators, which allows us to calculate them in the form of the ε expansion. In all known cases these dimensions turn out to be positive, i.e. the corresponding $\xi_n < 0$. This means that contributions in the form of powers of r/L are correction terms and the correlation functions possess finite limit at $r/L \to 0$. In the theory of turbulence $\xi_n > 0$ for all $n > 3$, from the point of view of the operator expansion it means that in the theory there are operators with negative critical dimensions ('dangerous operators'). Such operators have not been found yet. The problem is that it is difficult to identify them with the aid of the ε expansion due to the large real value of the parameter ε: there is a class of operators for which the ε expansion terminates [7], but there are no dangerous operators among them and the consistency of approximate calculation of the rest rises legitimate doubts[4].

The problem of calculation of characteristic quantities, for which the ε does not terminate, is common in the theory of turbulence, although there are known cases, when already the account of the first term of this expansion leads to fairly good agreement with experiment. The calculation of the turbulent Prandtl number—the ratio of the effective coefficient of the viscosity and the effective coefficient of the thermal diffusivity—may serve as an example [10] (the coinciding in the form result for equations of magnetohydrodynamics was obtained in [11]). It was unclear until recently how coincidental the agreement with the experiment is in this case, because the magnitude of the correction was not known: all calculations were restricted to the lowest one-loop approximation. The first two-loop calculation in the theory of turbulence was carried out for the Kolmogorov constant [6] and it showed that the two-loop contribution in this case is of the same order of magnitude as the one-loop contribution, thus confirming the pessimistic prognoses.

[4] The methods of the RG and OPE have been successfully used for the description of the anomalous scaling in the problem of turbulent mixing of a passive admixture. In the simplified model of Kraichnan the anomalous exponents of structure functions of the admixture have been calculated at the order ε^3 [8], and in a more realistic model (1), (27) (see below) at the order ε^2 [9].

In the present paper, we consider the problem of evaluation of the Kolmogorov constant and the turbulent Prandtl number without touching the delicacies of anomalous scaling. It will be shown that satisfactory results in calculation of these quantities may be obtained if a partial resummation of ε is carried out which removes singularities of its coefficients as functions of the space dimension d at $d \to 2$.

2. Kolmogorov constant, skewness factor

As we have seen, in the theory of turbulence the amplitude D_0 depends on the parameter ε. Only the asymptotic form of this dependence in the limit $\varepsilon \to 2$ dictated by the condition (11) is determined unambiguously. Obviously, the physical content of the theory is not changed, if the right-hand side of (11) is multiplied by any function $\psi(\varepsilon)$ such that $\psi|_{\varepsilon=2} = 1$. This means that at $\varepsilon \neq 2$ all quantities depending on D_0 acquire an arbitrary dependence on ε. The Kolmogorov constant C_{K} belongs to them, thus its ε expansion is not uniquely defined. This explains different values of C_{K} obtained by various authors in the one-loop approximation [12–20].

To obtain unambiguous results it is necessary to deal with universal quantities independent of D_0. These are quantities such as critical dimensions of the fields, parameters and composite operators, as well as suitable ratios of amplitudes, e.g. the turbulent Prandtl number. Other examples of such quantities are dimensionless ratios of the structure functions $S_n(r)\big/S_2^{n/2}(r)$, in particular, the skewness factor

$$\mathcal{S} \equiv S_3\big/S_2^{3/2}, \tag{14}$$

independent of the separation distance r in the Kolmogorov approximation (7). The real values of C_{K} and $\mathcal{S}$ are connected, according to (6), (7) and (14) by the relation

$$C_{\mathrm{K}} = \left[-\frac{12}{d(d+2)\mathcal{S}}\right]^{2/3}. \tag{15}$$

Having calculated part of the ε expansion of the skewness factor $\mathcal{S}$ and putting $\varepsilon = 2$ with the aid of (15) we may then find the corresponding approximation for the Kolmogorov constant C_{K}. In the evaluation of the skewness factor (14) there is no need to carry out the RG calculation of the function $S_3(r)$ because the exact result (10) may be used. There is, however, an additional problem in the calculation of $S_2(r)$ which may be circumvented if, instead of the function $S_2(r)$ its derivative $\tilde{S} \equiv r\partial_r S_2(r)$ is used [6]. In the inertial range the latter differs from $S_2(r)$ by the factor $2/3$ only at the real value $\varepsilon = 2$. From $\tilde{S}_2(r)$ and $S_3(r)$ the dimensionless ratio

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{(-S_3(r))^{2/3}} \tag{16}$$

may be constructed which, as the skewness factor, does not depend on the amplitude D_0, hence having a uniquely defined ε expansion. Having calculated first terms of this expansion, we may thus at the real $\varepsilon = 2$ find, with the use of (14) and (15), the skewness factor and the Kolmogorov constant:

$$\mathcal{S} = -\left[\frac{2}{3Q(2)}\right]^{3/2}, \qquad C_{\mathrm{K}} = \frac{3Q(2)}{2}\left[\frac{12}{d(d+2)}\right]^{2/3}. \tag{17}$$

The ε expansion of the quantity $Q(\varepsilon)$ is of the form

$$Q(\varepsilon, d) = \varepsilon^{1/3}\sum_{k=0}^{\infty} Q_k(d)\varepsilon^k, \tag{18}$$

whose coefficients $Q_k(d)$ are successively determined as the result of one-loop, two-loop etc calculations. The one-loop calculation is feasible in arbitrary space dimension [6],

$$Q_0 = (1/3)[4(d+2)]^{1/3}, \tag{19}$$

whereas a numerical two-loop calculation for $d = 3$ yields $Q_1/Q_0 \simeq 0.525$ [6]. Substitution of these results in (17) and (18) yields for the Kolmogorov constants the values $C_{\mathrm{K}} = 1.47$ and $C_{\mathrm{K}} = 3.02$ in the one-loop and the two-loop approximation, respectively, the former of them being smaller than the most reliable experimental estimate $C_{\mathrm{K}} = 2.01$ [21] and the latter being significantly larger. Thus, the account of the two-loop correction leads to the change of the one-loop value of C_{K} in the 'necessary direction', but the large relative magnitude of the correction (of the order 100%) gives rise to doubts about the consistency of the results in this calculational scheme.

In this context, we recall that ε expansions of various statistical characteristic quantities obtained in the framework of the RG method in the majority of field-theoretic models are semi-convergent, in which it is reasonable to keep some finite number of terms, because from a certain point the series begins to diverge drastically. In practice this is determined by the relative magnitude of the last term taken into account. In the present case, the first correction is already rather large, and in such a situation the necessity arises in summing—even if approximately—of the whole ε expansion. In the theory of critical phenomena the Borel summation of semi-convergent series is widely used to this end with the knowledge of both the first few terms of the ε expansion and the asymptotics of the expansion coefficients at large orders obtained with the aid of the 'instanton approach' [22]. In our case the prerequisites for such a summation are absent: only two terms of the expansion are known and only first steps of the instanton approach to dynamic problems have been made [23].

Nevertheless in the theory of turbulence it is possible to carry out a summation of the ε expansion with the use of specific features of the coefficients of this expansion as functions of the space dimension d [24]. Calculation of the two-loop contribution to the quantity (18) at $d \neq 3$ has shown that the relative portion of the two-loop contribution decreases with the growth of d and in the limit $d \to \infty$ it is only 10% of the one-loop contribution. At the same time this portion grows with decreasing d tending to infinity, when $d \to 2$ [6]. Such behaviour is a consequence of that the coefficients of the ε expansion have singularities at small d, the nearest of which lies at the point $d = 2$. The analysis of the diagrams of the perturbation theory shows that these singularities are poles in $d - 2 \equiv \Delta$, the order of which increases with the order of the perturbation expansion, so that the coefficients $Q_k(d)$ in (18) may be expressed in the form of the Laurent expansion

$$Q_k(d) = \sum_{l=0}^{\infty} q_{kl} \Delta^{l-k}, \qquad \Delta \equiv d - 2. \tag{20}$$

Substitution of (20) in (18) leads to the representation of the quantity $Q(\varepsilon, d)$ in the form of the double sum

$$Q(\varepsilon, d) = \varepsilon^{1/3} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (\varepsilon/\Delta)^k q_{kl} \Delta^l. \tag{21}$$

It turns out to be important that diagrams having poles at $d \to 2$ yield the main contribution in $Q_k(d)$ (of the order 90%) also at $d = 3$. It allows us to hope for a successful result if the leading singularities in $d - 2$ may be singled out and summed up in all orders of the ε expansion. This may be done by taking repeatedly advantage of the possibilities of the renormalization group.

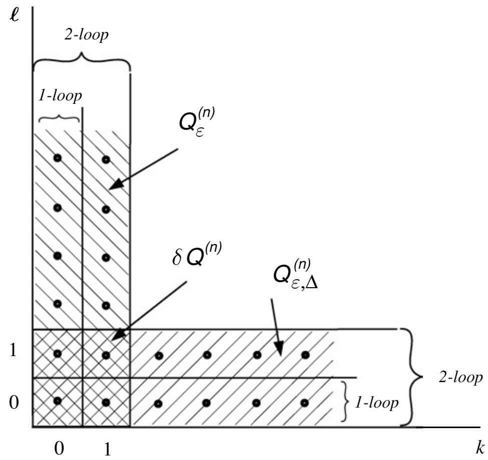

Figure 1. Illustration of the subsequences of the double sum (21) summed in the calculation of $Q_{\rm eff}^{(n)}$ in equation (26). Terms in the double sum (21) taken into account in $Q_{\varepsilon,\Delta}^{(n)}$ and $Q_{\varepsilon}^{(n)}$ correspond to the shaded horizontal and vertical stripes, respectively. The correction term $\delta Q^{(n)}$ corresponds to sum over the double-shaded square.

Setting the quantity $\Delta \equiv d-2$, an additional to ε formal small parameter of the theory of the same order with ε, we may renormalize the theory in the vicinity of $d=2$. This requires the introduction of an additional renormalization constant of the term $\sim\varphi'\varphi'$ in the action (8), which assumes a local form at $d=2$. Thus, a theory with two charges and two β functions emerges, and the one-loop calculation shows that it also has an IR-stable fixed point [25]. Physical quantities in this theory are constructed in the form of expansions in ε, the coefficients of which depend on the parameter $\xi \equiv \Delta/\varepsilon$:

$$Q(\varepsilon,\xi) = \sum_{k=0}^{\infty} \Psi_k(\xi)\varepsilon^k, \qquad \xi \equiv \Delta/\varepsilon. \tag{22}$$

Comparing (22) with (21) we infer

$$\Psi_k(\xi) = \sum_{l=0}^{\infty} q_{lk}\xi^{k-l}. \tag{23}$$

Thus, there are two alternative ε expansions (16) and (22). The corresponding n-loop calculations in the usual renormalization scheme and in the scheme of double expansion in the vicinity of $d=2$ allow us to find the initial terms of these expansions

$$Q_{\varepsilon}^{(n)} \equiv \sum_{k=0}^{n-1} Q_k(d)\varepsilon^k, \tag{24}$$

$$Q_{\varepsilon,\Delta}^{(n)} \equiv \sum_{k=0}^{n-1} \Psi_k(\xi)\varepsilon^k. \tag{25}$$

Relations (24) and (25) contain complementary information, which becomes obvious in figure 1 illustrating infinite subsequences of the double sum (21) being summed up in (24) and (25). A term in the sum (21) is depicted by a dot (k,l) in figure 1. An exact expression for $Q(\varepsilon,d)$ is obtained by the account of all dots from the first quadrant. The shaded regions in

Table 1. One- and two-loop values of the Kolmogorov constant in the usual ε expansion (C_ε) and the double ε, Δ expansion ($C_{\varepsilon,\Delta}$); the contribution C_δ from the correction $\delta Q^{(n)}$ in equation (26), and the value C_{eff} from equation (26).

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	C_{eff}
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

figure 1 correspond to dots taken into account in the two-loop approximation: the horizontal stripe corresponds to terms from $Q_\varepsilon^{(n)}$ (24), the vertical stripe to those from $Q_{\varepsilon,\Delta}^{(n)}$ (25). All terms in the shaded area will be taken into account in the effective quantity

$$Q_{\text{eff}}^{(n)} = Q_\varepsilon^{(n)} + Q_{\varepsilon,\Delta}^{(n)} - \delta Q^{(n)}, \tag{26}$$

where

$$\delta Q^{(n)} \equiv \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}(\varepsilon/\Delta)^k q_{kl}\Delta^l$$

is a subtraction term necessary to avoid double counting of terms with $k \leqslant n-1, l \leqslant n-1$ (the double-shaded square in figure 1). It may be found by taking the corresponding number of terms from expansions (20) or (22). From the point of view of the usual ε expansion (16) relation (26) may be interpreted as follows: in the $n-1$ first terms of the expansion the coefficients $Q_k(d)$ from equation (16) are calculated exactly, but in all higher-order terms ($k \geqslant n$) approximately with the account of $n-1$ first terms of their Laurent expansion (20).

In table 1 we have quoted for comparison the values of the Kolmogorov constant calculated at first and second order of the usual ε expansion (C_ε), the double ε, Δ expansion ($C_{\varepsilon,\Delta}$), the contribution C_δ from the correction $\delta Q^{(n)}$ in equation (26) and the value C_{eff} obtained from the relation (26). In all the cases quoted, the recommended experimental value of the Kolmogorov constant $C_{\text{exp}} = 2.01$ lies between the values of the first and second approximations. However, the difference between these values is rather significant both in the ε expansion and in the (ε, Δ) expansion, let alone the leading terms of the ε expansion of the latter. For the improved ε expansion, i.e. for the quantity $C_{\text{eff}} = C_\varepsilon + C_{\varepsilon,\Delta} - C_\delta$ calculated according to equations (26), however, this difference is about three times smaller leading to far better agreement with the experimental data.

3. Turbulent Prandtl number

The idea of improving the ε expansion described above has been checked in the problem of description of the process of the turbulent mixing of a passive admixture [26]. This process is described by the diffusion equation with the convective transport term:

$$\partial_t\psi + (\varphi_j\partial_j)\psi = \kappa_0\Delta\psi + f. \tag{27}$$

The field $\psi(\mathbf{x}, t)$ in equation (27) may have the meaning of both the non-uniform temperature (κ_0 being the coefficient of thermal diffusivity) and concentration of the particles of the admixture (in this case κ_0 is replaced by the coefficient of diffusion). The field $f(\mathbf{x}, t)$ is the source of the passive scalar field. The passiveness of the field ψ shows in that it does not affect the correlation functions of the field of turbulent eddies φ.

As the object of calculation in the model (27) we chose the turbulent Prandl number. Let us recall that the Prandtl number Pr is the dimensionless ratio of the coefficient of kinematic viscosity ν_0 to the coefficient of thermal diffusivity κ_0: $Pr = \nu_0/\kappa_0$. (In the formally identical problem of turbulent diffusion the ratio of the coefficients of kinematic viscosity and diffusion is called the Schmidt number). For systems with strongly developed turbulence the process of homogenization of the temperature is strongly accelerated, which is reflected in the value of the effective or turbulent coefficient of thermal diffusivity $\kappa_{\rm tur}$. The ratio of the coefficient of turbulent viscosity $\nu_{\rm tur}$ and the coefficient of turbulent thermal diffusivity is the turbulent Prandtl number: $Pr_{\rm tur} = \nu_{\rm tur}/\kappa_{\rm tur}$. Contrary to its molecular analogue the turbulent Prandtl number is universal, i.e. does not depend on individual properties of the fluid.

The stochastic problem (1), (2), (27) is equivalent to the quantum-field model with the doubled number of fields $\Phi \equiv \{\varphi, \psi, \varphi', \psi'\}$ and the action

$$S(\Phi) = \varphi' D_F \varphi'/2 + \varphi'[-\partial_t\varphi + \nu_0\Delta\varphi - (\varphi\partial)\varphi] + \psi'[-\partial_t\psi + \kappa_0\Delta\psi - (\varphi\partial)\psi + f]. \tag{28}$$

Let us specify the definition of the turbulent Prandtl number $Pr_{\rm tur}$ in the model (28) with the use of the Dyson equation

$$G^{-1}_{\psi\psi'}(k,\omega) = -{\rm i}\omega + \kappa_0 k^2 - \Sigma_{\psi'\psi}(k,\omega) \tag{29}$$

for the response function of the admixture field

$$G_{\psi\psi'}(\mathbf{x}-\mathbf{x}', t-t') \equiv \langle\psi(\mathbf{x},t)\psi'(\mathbf{x}',t')\rangle|_{f=0} = \left.\frac{\delta\langle\psi(\mathbf{x},t)\rangle}{\delta f(\mathbf{x}',t')}\right|_{f=0} \tag{30}$$

and the analogous equation for the eddy field φ

$$G^{-1}_{\varphi\varphi'}(k,\omega) = -{\rm i}\omega + \nu_0 k^2 - \Sigma_{\varphi'\varphi}(k,\omega) \tag{31}$$

(Σ are the self-energy operators). Proceeding from (29) and (31) we introduce effective low-frequency coefficients of viscosity $\nu_{\rm tur}$, thermal diffusivity $\kappa_{\rm tur}$ and the corresponding effective Prandtl number $Pr_{\rm tur}$

$$\begin{aligned} &\nu_{\rm tur} \equiv \nu_0 - \Sigma_{\varphi'\varphi}(k,\omega=0)/k^2, \\ &\kappa_{\rm tur} \equiv \kappa_0 - \Sigma_{\psi'\psi}(k,\omega=0)/k^2, \qquad Pr_{\rm tur} \equiv \nu_{\rm tur}/\kappa_{\rm tur}. \end{aligned} \tag{32}$$

The relation (32) for $Pr_{\rm tur}$ reduces to the usual definition of the Prandtl number for a laminar flow (at $\Sigma = 0$); for turbulent systems $Pr_{\rm tur}$ is independent of the wavenumber k in the inertial range.

Renomalization of the model (28) requires the introduction of an additional renormalization constant of the coefficient of thermal diffusivity κ_0: $\kappa_0 = Z_\kappa \kappa$. Calculation of this constant has shown that the extended model also has an IR-stable fixed point [10]. This is the basis for the calculation of the turbulent Prandtl number (32) in the form of the ε expansion. The two-loop calculation at $d = 3$ yields [26]

$$Pr_{\rm tur} = Pr^{(0)}_*(1 + 0.0358\varepsilon) + O(\varepsilon^2), \qquad 1/Pr^{(0)}_* = \frac{\sqrt{43/3}-1}{2} \simeq 1.3930. \tag{33}$$

At the physical value $\varepsilon = 2$ this yields for the turbulent Prandtl number ${\rm Pr}_t$ the result,

$$Pr^{(0)}_{\rm tur} \simeq 0.72, \qquad Pr_{\rm tur} \simeq 0.77 \tag{34}$$

in one-loop and two-loop accuracy, respectively. These values are in a very good agreement with the experimental estimate $Pr_{\rm tur} \simeq 0.81$ currently considered the most reliable [27–29], and the account of the two-loop correction notably improves the agreement with the experiment.

The small value obtained for the two-loop contribution to the turbulent Prandtl number appears quite astonishing: like in the calculation of the Kolmogorov constant separate diagrams yield rather large contributions to the coefficients $\nu_{\rm tur}$ and $\kappa_{\rm tur}$, while in the ratio $\nu_{\rm tur}/\kappa_{\rm tur}$ these contributions compensate each other nearly completely. The reason is clarified by inspection of the dependence of these contributions on the space dimension d. It turns out that—as in the case of the Kolmogorov constant—large contributions at $d = 3$ are produced by diagrams having a pole in $\Delta = d - 2$. In the ratio $\nu_{\rm tur}/\kappa_{\rm tur}$, however, these pole contributions are completely mutually cancelled.

Thus, our results for the turbulent Prandtl number complement the conclusion made for the Kolmogorov constant. In the two-loop approximation the main contribution is due to graphs having a singularity at $d = 2$ and it is necessary to sum such graphs. For quantities in which this singularity is absent, the two-loop contribution is relatively small and the results of the usual ε expansion appear fairly reliable at the level of accuracy suggested by the two-loop correction.

4. Conclusion

The analysis carried out here allows us to draw two main conclusions about the possibilities of the use of the ε expansion in the stochastic theory of turbulence.

(i) This expansion is well defined only for universal quantities independent of the amplitude of the correlation function of random force.

(ii) To obtain acceptable quantitative results in calculation of those physical quantities, whose ε expansion does not terminate, it is necessary to analyse the behaviour of the coefficients of the expansion in the vicinity of $d = 2$. If they have finite limit, when $d \to 2$, then we may expect a successful description of this particular physical quantity (at $d = 3$) with the account of first terms of its ε expansion (as in the case of the turbulent Prandtl number). If, on the contrary, the coefficients of the expansion of the physical quantity have poles at $d \to 2$, then it is necessary to sum the pole contributions in all orders of the ε expansion and for the calculation of this quantity use the 'improved ε expansion'.

In conclusion, let us point out that the procedure of construction of the improved ε expansion described here is relevant for three-dimensional systems. For two-dimensional turbulence separate analysis is needed due to both its specific physical properties (the presence of an infinite number of conservation laws, the inverse energy cascade) and purely technical problems: the necessity to take into account singularities of the expansion coefficients in even lower dimensions of space d.

Acknowledgments

L Ts Adzhemyan, A N Vasil'ev, T L Kim and M V Kompaniets thank for support from the Russian Foundation for Basic Research (grant no 05-02-17524). A N Vasil'ev thanks the Russian National Program for support (grant no 2. 1. 1. 1112).

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